From Yang-Mills lagrangian to MHV diagrams

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Abstract: We prove the equivalence of a recently suggested MHV-formalism to the standard Yang-Mills theory. This is achieved by a formally non-local change of variables. In this note we present the explicit formulas while the detailed proofs are postponed to a future publication.

Contents

1. Introduction

Recently, a new approach to the perturbative calculations in Yang-Mills (YM) theory has been suggested by Cachazo, Svrček and Witten (CSW) [1]. In this new formalism, the vertices are obtained from the so-called MHV-amplitudes (i.e. the amplitudes maximally violating the helicity) by a suitable continuation off shell. This technique was shown to reproduce all known gluon tree amplitudes and predicts a number of new results [2]. The successful generalization for the one-loop amplitudes has been also developed [3] although a new additional vertex has to be added at one-loop level in YM theory without supersymmetry. The MHV-like diagrams for the gravity case have been formulated as well [4]. A complete list of references can be found in [5].

In this paper we address the question of equivalence between the MHV diagrams and the conventional YM perturbation theory expansion. The MHV diagram rules can of course be described with help of an action functional, which we call the CSW action. It turns out that there exists a change of variables transforming the standard YM action to the CSW action. The formula for such a change of variables is obtained as follows. First, we recall a certain solution to the self-duality equation which serves for a swift derivation of the MHV-amplitudes [6, 7]. This self-dual gauge field can be continued off shell in the spirit of ref. [1] and provides very explicit change of variables which brings YM lagrangian in the light-cone gauge into the form of CSW lagrangian. At present, we can check this by a brute-force calculation only and feel that a better, more conceptual understanding of our result is needed. This is despite the fact that the formula for the change of variables is perfectly explicit and the geometrical origin of the self-dual solution behind it seems to be well understood. Therefore, we give here those explicit formulas and postpone the detailed proofs to a future publication.

The paper is organized as follows. First, we remind the MHV diagram rules (section 2) and present the YM action in the light-cone gauge (section 3). Then, in section 4 we describe a solution to the self-duality equation which is relevant to the MHV-amplitudes. A change of variables in the light-cone YM action, which renders it to the CSW action, is introduced in section 5. Some open questions are mentioned in the concluding section.

2. MHV diagrams

Let us remind main points concerning MHV diagrams. The basic ingredient is MHV $(--, + \ldots +)$ amplitude describing the tree scattering of two gluons of negative helicity and arbitrary number of positive helicity gluons. The amplitude turns out to be a simple rational function of on-shell momenta of massless particles and reads as [8, 9]

$$A(1^-, 2^-, 3^+ \ldots, n^+) = g^{n-2} \frac{\langle 1,2 \rangle^4}{\langle 1,2 \rangle \langle 2,3 \rangle \ldots \langle n,1 \rangle} \tag{2.1}$$

where the on-shell momentum of massless particle in the standard spinor notations reads as $p_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$, λ_a and $\tilde{\lambda}_{\dot{a}}$ are positive and negative helicity spinors. Inner products in spinor notations read as $\langle \lambda_1, \lambda_2 \rangle = \epsilon_{ab} \lambda_1^a \lambda_2^b = \langle 1,2 \rangle$ and $[\tilde{\lambda}_1, \tilde{\lambda}_2] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_1^{\dot{a}} \tilde{\lambda}_2^{\dot{b}}$. These amplitudes were interpreted as correlators in the auxiliary two-dimensional theory in [10] and in terms of topological string on twistor target space in [11]. There are no amplitudes with zero or one negative helicity gluons at the tree level however these amplitudes emerge at one-loop level in the YM theory without supersymmetry [12]. For instance, one-loop all-plus amplitude reads as

$$A^{one-loop}(+, \ldots, +) = g^n \sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq n} \frac{\langle i_1, i_2 \rangle [i_2, i_3] \langle i_3, i_4 \rangle [i_4, i_1]}{\langle 1,2 \rangle \langle 2,3 \rangle \ldots \langle n,1 \rangle} \tag{2.2}$$

It was suggested in [1] that conventional YM diagrams in both supersymmetric and non-supersymmetric gauge theories can be reorganized in the different way which nowadays is known as MHV diagrams or CSW lagrangian. The building blocks of this diagrammatics are MHV vertices extended off-shell and the canonical propagator $\frac{1}{P^2}$ involving $(+-)$ degrees of freedom and connecting two MHV vertices. The continuation off-shell suggested in [1] for λ in any internal line reads as

$$\lambda_a = p_{a,\dot{a}} \eta^{\dot{a}} \tag{2.3}$$

where η is arbitrary spinor fixed for off-shell lines in all diagrams relevant for a given amplitude.

At higher loops the situation turns out to be more subtle at least in the theory without supersymmetry. The non-vanishing all-plus one-loop amplitude can not be derived from MHV vertices only that is why it was suggested in [3] that one-loop all-plus diagram has to be added to the CSW lagrangian as a new vertex. It was also argued that there is no need to add one-loop vertex $(-, + \ldots +)$ to new lagrangian. The situation in SUSY case is more safe since these amplitudes vanish however even in this case it is unclear if new vertices have to be added to reproduce higher loops results.

In spite of the considerable success of this approach its conceptual origin remained obscure and it was unclear how these effective degrees of freedom involved into the CSW lagrangian are related with the conventional YM gauge fields. It is the goal of this paper to argue that these effective degrees of freedom emerge from the standard YM variables in the light-cone gauge upon the particular "dressing" procedure.

3. Yang-Mills on the light cone

In this section we briefly discuss YM theory in the light-cone gauge which involves only two physical degrees of freedom. The lagrangian of YM theory in the light-cone variables has been found in N=4 SUSY case [13, 14]. In what follows we shall exploit Mandelstam two-field formulation [13] which has been successfully used recently in one-loop calculations in YM theories with different amount of supersymmetry [15]. Two fields Φ_+ and Φ_- are related with the physical transverse degrees of freedom of the gluon as follows

$$\Phi_-(x) = \partial_+^{-1} A(x), \quad \Phi_+(x) = \partial_+ \bar{A}(x) \tag{3.1}$$

We shall be interested in the non-supersymmetric theory with the action in $A_+ = 0$ gauge

$$S = \int d^4 x [\Phi_+^a \Box \Phi_-^a + 2g f^{abc} \partial_+ \Phi_-^a \bar{\partial} \Phi_-^b \Phi_+^c + 2g f^{abc} \partial_+^2 \Phi_-^a \partial_+^{-2} \partial \Phi_+^b \partial_+^{-1} \Phi_+^c$$

$$- 2g^2 f^{abc} f^{ade} \partial_+^{-2} (\partial_+ \Phi_-^b \Phi_+^c)(\partial_+^{-1} \Phi_+^d \partial_+^2 \Phi_-^e)] \tag{3.2}$$

where $\partial = \frac{1}{\sqrt{2}}(\partial_{x_1} + i\partial_{x_2})$ is derivative with respect to the transverse coordinates x_1, x_2 and $\bar{\partial} = \partial^*$. The action contains local and non-local triple vertices as well as non-local quartic vertex.

Let us make a few comments on the form of the action (3.2). First note that it involves two fields of dimensions 0 and 2 hence positive and negative helicity fields enter lagrangian asymmetrically. In particular, vertex $(-++)$ is local in the coordinate space while $(--+)$ is not. There are two classes of solutions to the equations of motion which correspond to the self-duality and anti-self-duality equations written in a little bit unusual form, namely

$$\Phi_- = 0 \qquad \Box \Phi_+ = (\Phi_+, \Phi_+) \tag{3.3}$$

and

$$\Phi_+ = 0 \qquad \Box \Phi_- = \{\Phi_-, \Phi_-\} \tag{3.4}$$

where the schematically written r.h.s. are obtained by the variations of the cubic terms in the action (3.2).

Note that the truncation of the light-cone action to the first two terms which amounts to the self-dual equation of motion for the negative helicity field has been discussed in the context of MHV amplitudes in [17]. However this truncation evidently can not be equivalent to the full YM theory we are dealing with. The action (3.2) is gaussian with respect to both fields that is one of them can be integrated out yielding highly nontrivial effective action with a non-canonical kinetic term for the other.

4. MHV vertex from self-duality equation

The essential ingredient of the MHV diagrams is MHV vertex and in this section we shall argue that solution to the self-duality equation with the particular boundary conditions serves as the generating function for all MHV amplitudes. This fact has been recognized some time ago by Bardeen [6] and has been elaborated further in [7, 16 – 18]. In what follows just this solution to the self-duality equation provides the desired change of variables from conventional YM to MHV formalisms.

Let us briefly remind the derivation of the perturbiner solution to the self-duality equation following [7]. The self-dual perturbiner yields the form-factor of the one off-shell gluon between the vacuum and arbitrary number of gluons of the same helicity, momenta p_j and color orientations t_j. The starting point is the transition to the twistor representation with the additional spinor homogeneous coordinate ρ^α on the auxiliary CP^1. The self-duality equation in the twistor representation is equivalent to the zero-curvature condition

$$[\nabla_{\dot\alpha} \nabla_{\dot\beta}] = 0 \tag{4.1}$$

where $\nabla_{\dot\alpha} = \rho^\alpha \nabla_{\dot\alpha,\alpha}$. Hence the solution to the self-duality equation can be represented in the following form

$$A_{\dot\alpha} = g^{-1} \partial_{\dot\alpha} g \tag{4.2}$$

where $\partial_{\dot\alpha} = \rho^\alpha \partial_{\dot\alpha,\alpha}$, $A_{\dot\alpha} = \rho^\alpha A_{\dot\alpha,\alpha}$ and g is group valued function depending on ρ and x, as well as on the quantum numbers p_j and t_j of the external particles. We assume that $A_{\dot\alpha}$ is a polynomial of degree one in ρ. Then the group element necessarily has to be meromorphic function of ρ of degree zero such that connection $A_{\dot\alpha}$ is regular at the poles.

The perturbiner is defined as a solution to the self-duality equation of the shape of a formal expansion in the (non-commuting) variables $E_j = t_j e^{ip_j x}$, which are essentially the plane waves of the external gluons of the same, say positive, helicity. That is we look for the group element providing the solution to the zero curvature equation in the following form

$$g_{ptb}(\rho) = 1 + \sum_j g_j(\rho) E_j + \cdots + \sum_{j_1 \ldots j_L} g_{j_1 \ldots j_L}(\rho) E_{j_1} \ldots E_{j_L} + \cdots , \tag{4.3}$$

where different terms with L of E's correspond to different color orderings in the form-factors with L external particles. The regularity of the connection at the poles leads us immediately to a unique solution for coefficients of the expansion of $g_{ptb}(\rho)$ [7]:

$$g_{j_1 \ldots j_L}(\rho) = \frac{\langle \rho, q \rangle}{\langle \rho, j_1 \rangle \langle j_1, j_2 \rangle \langle j_2, j_3 \rangle \ldots \langle j_{L-1}, j_L \rangle} \tag{4.4}$$

where the so-called reference spinor q_α is the one which enters into the polarization vectors $\epsilon^j_{\dot\alpha,\alpha} = q_\alpha \tilde\lambda^j_{\dot\alpha}$.

The corresponding connection A_{ptb} can be found upon the substitution of the solution into (4.2). Let us note that perturbiner solution itself is localized on the line in the twistor space if one performs half-Fourier transforms for all massless particles involved in the form-factor similar to [11].

The perturbiner solution describes form-factor or off-shell current of the form $\langle A_{\dot{\alpha},\alpha}(k)\rangle_{k_1,...,k_n}$ where the gluon with momentum k is off-shell while all other gluons are on-shell and have the same helicity. Using the explicit form of the solution one can verify that this form-factor has no pole in k^2 and, hence, gives zero upon the application of the reduction formula. This corresponds to the vanishing of the amplitude with all but one gluons of the same helicity.

To get MHV amplitudes from the perturbiner solution one has to consider the linearized YM equation in the background of the perturbiner. The most compact form of the generating function for MHV amplitudes has the following structure [7]

$$M(k_1, k_2) = \langle 1, 2 \rangle^2 \int d^4 x Tr[E_1 g_{ptb}^{-1} E_2 g_{ptb}], \tag{4.5}$$

where k_1, k_2 are momenta of the negative helicity gluons and plane waves corresponding to the positive helicity gluons are substituted into g_{ptb}. The group elements which depend on the twistor variable have to be taken at points ρ_i corresponding to the momenta of negative helicity gluons.

5. Change of variables

Let us turn to the central point of our paper and describe the proper non-local change of variables. First, let us comment on the choice we shall make in a moment. In the light-cone action there is nontrivial $(-++)$ vertex which has to be absent in the CSW lagrangian. That is change of variables has to provide the removal of this term. It enters the equation of motion for Φ_+ which reduces to the self-duality equation if $\Phi_- = 0$. Hence we expect that change of variables we are looking for should map self-duality equation to the Laplace equation. Actually perturbiner solution to the self-duality equation does this job.

To describe new variables precisely let us represent combination involved in the equation of motion for the light-cone variable Φ_+ at $\Phi_- = 0$ in the form

$$\Box \Phi_+ + (\Phi_+, \Phi_+) = \partial_+ F'(\phi_+, \partial_+^{-1} \Box \phi_+) \tag{5.1}$$

where the following notation is assumed $\delta F = F'(\phi, \delta\phi)$. Now introduce new variable ϕ_+ by

$$\Phi_+ = F(\phi_+) \tag{5.2}$$

The convenient choice for the second field ϕ_- is dictated by the canonicity of the $(+-)$ propagator in new variables which yields

$$\Phi_- = \partial_+^{-4} F'(\phi_+, \partial_+^4 \phi_-) \tag{5.3}$$

The correct form of the propagator can be checked with the help of relation

$$\int d^4 x Tr[F'(\phi, v) \partial_+^{-3} F'(\phi, u)] = \int d^4 x Tr[u \partial_+^{-3} v] \tag{5.4}$$

valid for arbitrary u and v.

Now we are ready to make a link with the previous section. It turns out that

$$F(\phi_+) = \partial_+ g_{ptb}^{-1}(\phi_+) \partial g_{ptb}(\phi_+) \tag{5.5}$$

where group element in (5.5) is effectively continued off-shell. That is the off-shell field ϕ_+ is considered in g_{ptb} instead of the plane wave and momenta of plane waves are substituted by the corresponding derivatives

$$\Phi_+ = \partial_+^2 \sum_{n \geq 1} \frac{1}{\partial_{+,1} \langle \bar{\partial}_1 \bar{\partial}_2 \rangle \ldots \partial_{+,n}} \underbrace{\phi_+ \ldots \phi_+}_{n} \tag{5.6}$$

where $\bar{\partial}_k = \partial_{\alpha,k}$ acts on the k-th term in the product. Effectively the change of variables above kills $(++-)$ vertex in the action and maps solution to the self-duality equation to the solution to the free Laplace equation. The rest of the check concerns the interaction part of the lagrangian. We have verified that $(--+)$ and $(++--)$ vertices in the light-cone lagrangian get combined together into the correct interaction terms in CSW lagrangian. That is we have argued that change of variables from light-cone YM fields yields at the tree level both correct propagator and vertices in the CSW lagrangian. Hence just fields $\phi_- \phi_+$ play the role of twistor degrees of freedom corresponding to positive and negative helicities. The technical details concerning the change of variables shall be presented elsewhere.

Let us make a few comments on the one-loop extension of the CSW lagrangian. As we have already mentioned in the non-supersymmetric case it has to be extended by all-plus one-loop amplitude. The possible origin of such correction is clear in our approach - there is jacobian of the change of variables. We have not proved that jacobian reproduces the desired answer but there are several arguments favoring this possibility. Naively the change of variables discussed is expected to be canonical that is if we would work with the system with finite number degrees of freedom then it would equal to one. However the theory at hands enjoys infinite dimensional phase space hence one could expect the anomalous jacobian of the canonical transformations. Second argument concerns the form of the naive jacobian which involves only powers of ϕ_+ as expected. Moreover if the additional one-loop term follows from the jacobian indeed then the absence of such terms in SUSY case could be attributed naturally to the standard SUSY cancellations.

Note that in principle the second similar change of variables can be done which would kill the $(--+)$ vertex as well. Upon this change the action would involve only quartic and higher interaction terms however the possible usefulness of such action is unclear to us at present.

6. Discussion

In this short note we questioned the relation between the conventional YM variables and effective degrees of freedom in the CSW lagrangian. The answer turns out to be remarkably simple - they are related just by the non-local change of variables. Tree diagrams are perfectly reproduced upon this change while the evident candidate for the one-loop completion of the action is the corresponding jacobian. Moreover our consideration implies that one should not expect additional terms in the CSW lagrangian at higher loops.

The immediate question in our approach concerns the twistor interpretation of the suggested change of variables. We expect that the interpretation of the perturbative YM theory in terms of the string on the twistor manifold [11] matches the twistor interpretation of the perturbiner developed in [7]. Of course the most interesting question raised in [6] concerning the expected relation with some hidden integrability-like structure responsible for the nullification of the infinite number of tree amplitudes remains open. Nevertheless we believe that our work could be useful for this line of reasoning.

There are several possible generalizations. First, supersymmetric case can be considered along this way starting with the corresponding light-cone formulation [13]. The perturbiner solution in the supersymmetric case has been found in [19]. It is known as well for the gravity case [20]. One more line of generalization concerns QED with massless or massive fermions. Recently MHV-type technique for tree QED was developed [21] which naturally captures the soft photons limit while the example of the stringy picture for MHV QED amplitude has been found in [22]. The change of variables could be found in this case as well and we expect that effective fermionic fields in MHV formulation of QED involve original fermions dressed by the infinite number of positive or negative helicity photons.

Acknowledgments

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